Journal of Statistical Physics, Vol. 57, Nos. 3/4, 1989

Mixing Properties of Quantum Systems

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Received October 22, 1988

We generalize the classical notion of topological mixing for automorphisms of C^*-algebras in two ways. We show that for Galilean-invariant Fermi systems the weaker form of mixing is satisfied. With some additional requirement on the range of the interaction we can also demonstrate the stronger mixing property.

KEY WORDS:

1. INTRODUCTION

Boltzmann's vision of the ergodic behavior of large systems is realized for asymptotically Abelian quantum systems. For these systems additional constants which might spoil ergodicity are in the center $\mathscr{Z}$ of the algebra of observables and if the system is completely quantal in the sense that $\mathscr{Z}$ is trivial, nothing obstructs ergodicity. In this paper we shall characterize a class of systems for which these conditions are met and show that they even have some strong mixing properties. For recent work on this problem see refs. 1–4.

Finite quantum systems are never mixing and since infinite systems might have finite parts which are shielded from the rest of the world, also infinite systems will, in general, not mix. However, if a system does not have any fixed part, such as particles interacting with potentials depending only on the distance between them, then one feels that any disturbance will eventually diffuse to infinity and there should be some mixing. We shall substantiate this feeling and show that under these circumstances the system mixes in the following way: For any two projectors (= propositions) P and Q and any $\varepsilon > 0$ there is a time T such that P and Q_t for all $t > T$ have almost a common eigenvector ψ. Almost is measured by ε in the sense $|\langle P\psi | Q_t \psi \rangle| > 1 - \varepsilon$ if P, $Q \neq 0$. A typical case is the proposition

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0022-4715/89/1100-0811$06.00/0

"there is a fermion in state f," where the corresponding projector is $n_f = \alpha_f^* \alpha_f$. The negation of it, "there is no fermion in state f," has the projector $1 - n_f$ and $n_f((1 - n_f) = 0$. Under the free time evolution n_f changes to n_{f_t} where $\langle f | f_t \rangle \to 0$. Thus, $n_f(1 - n_{f_t})$ will become $\neq 0$, in fact, for $\langle f | f_t \rangle = 0$ a vector with a particle in state f is a common eigenvector with eigenvalue 1 for n_f and $1 - n_{f_t}$. Our result extends this to all propositions of the observable algebra to a reasonable class of time evolutions. This means that the system is totally chaotic and all information gets lost since all propositions eventually become compatible. It seems that this is showing too much, since after all some information remains even in large quantum systems. For instance, if we have a mixture of two kinds of substances which do not react chemically, whatever happens we know what their densities are. The latter statement is correct and the formalism tells us that these densities are elements of $\mathscr{Z}$ and therefore pure numbers. They do not change with time and the product of any proposition associated with them is 0 or 1. Our results depends on the algebra of observables which are localized quantities. It does not apply to global quantities constructed with strong limiting procedures, For instance, in the vacuum representation e^{iHt} can be constructed as strong limit of local observables, and since this operator is constant in time, its projections obviously do not satisfy our condition.

Finite quantum systems are never completely mixed through, since there is at least one nontrivial constant, namely the Hamiltonian, and different energy shells do not communicate. However, infinite systems should be considered as open, since only finite parts are observable and they always interact with the outside. Hence, the Hamiltonian is not part of the observables and it is possible that the multiples of unity are the only constant observables. Our result shows that under these circumstances the quantum systems have strong mixing properties, whereas classically they are, in general, only ergodic.

Considering spatially infinitely extended systems, one faces the problem of whether a formal Hamiltonian defines a time evolution. For the free time evolution or for some lattice systems with finite-range interaction we know this to be the case. For the continuous systems we are interested in the problem is nontrivial, since the usual perturbation theory with respect to the potential cannot converge. This is suggested by Dyson's old argument that a convergent perturbation expansion would define the time evolution also for the potential with opposite sign. However, there is no potential known for which the system is stable for both signs of the potential and for one of the two signs a catastrophic behavior is to be expected. Some results on this question have been obtained by various authors.[5,6] We shall not pursue it further, but hope for the best.

2. THE MIXING THEOREM

The observables of a quantum system usually form a C^*-algebra $\mathscr{A}$. We shall be concerned with an infinite Fermi system where the field algebra $\mathscr{F}$ consists of polynomials in the creation operators

$$\alpha_f^* = \int d^3x f(x)\, \alpha^*(x) = \int \frac{d^3k}{(2\pi)^3} \tilde{f}(k)\, \tilde{\alpha}^*(k)$$

the destruction operator α_g, and norm limits thereof. The wave functions f and g are from $L^2(\mathbf{R}^3)$, since $\|\alpha_g\|^2 = \|g\|^2 = \int d^3x\, |g(x)|^2$. Here $\mathscr{A}$ is the subalgebra of $\mathscr{F}$ where each term in the sum has an equal number of α's and α^*'s. A state ϕ over $\mathscr{F}$ is a positive linear functional over $\mathscr{F}$, that is, a linear map $\mathscr{F} \to C$ such that $\mathscr{F}^+ \to \mathbf{R}^+$, $\mathscr{F}^+$ denoting the positive elements from $\mathscr{F}$. The GNS construction associates to ϕ via the regular representation of $\mathscr{F}$ a representation Π_ϕ in a Hilbert space $\mathscr{H}_\phi$. For instance, the thermal state with the two-point function

$$\phi_\beta(\alpha_f^* \alpha_g) = \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{f}(k)\, \tilde{g}^*(k)}{1 + \exp[\beta(k^2 - \mu)]} \tag{2.1}$$

and vanishing reduced ($n \geqslant 3$)-point functions give a Fock representation in terms of two Fermi fields $\alpha_{1,2}$:

$$\Pi_{\phi_\beta}(\tilde{\alpha}(k)) = \frac{\tilde{\alpha}_1(k)}{\{1 + \exp[-\beta(k^2 - \mu)]\}^{1/2}} + (-)^{N_1} \frac{\tilde{\alpha}_2^*(k)}{\{1 + \exp[\beta(k^2 - \mu)]\}^{1/2}} \tag{2.2}$$

The Fock vacuum $|0\rangle$, $\tilde{\alpha}_1(k)\, |0\rangle = \tilde{\alpha}_2(k)\, |0\rangle = 0$ reproduces ϕ: $\langle 0|\, \Pi_{\phi_\beta}(a)\, |0\rangle = \phi_\beta(a)$ $\forall a \in \mathscr{F}$. These representations are reducible, the commutant $\mathscr{F}' = \{a \in \mathscr{B}(\mathscr{H}_\phi)$: $[a, b] = 0$, $\forall b \in \Pi_\phi(\mathscr{F})\}$ is nontrivial, in fact, conjugate-isomorphic to $\mathscr{F}$. The following features will be important:

1. ϕ_β is faithful: $\phi_\beta(a) = 0 \Leftrightarrow a = 0$, $\forall a \in \mathscr{F}^+$, $\beta \neq \infty$.
2. Π_{ϕ_β} are factor representations, $\mathscr{Z} \equiv \mathscr{F}' \cap \mathscr{F}'' = c\mathbf{1}$.
3. The infinite-temperature state ϕ_0 is tracial, $\phi(ab) = \phi(ba)$.
4. The Fock state corresponds to $\beta = \infty$, $\phi_\infty(\alpha^* \alpha) = 0$. The corresponding representation is irreducible.

Remark 2.1. For Abelian algebras all states are tracial, whereas for the $(n \times n)$ matrices M_n only $(1/n)$ tr is a tracial state. $\mathscr{F}$ belongs to a class of algebras, the so-called UHF or Glimm algebras, for which there is exactly one faithful tracial state ϕ_0; as a consequence ϕ_0 is invariant under any automorphism τ of $\mathscr{F}$, since $\phi_0 \circ \tau$ is also tracial and therefore ϕ_0. Furthermore, Π_{ϕ_0} is a factor representation, since $\forall z \in \mathscr{Z}^+$, $\phi_0'(a) = \phi_0(za)$ would also be tracial and $\phi_0(za) = \phi_0(a)\ \forall a$ implies $z = 1$.

The local structure of $\mathscr{A}$ suggests that macroscopically separated parts should be independent in the sense that the corresponding observables commute. These ideas are made precise by the following definition.

Definition 2.2. Let τ_t be a one-parameter group of $\mathscr{A} \to \mathscr{A}$, $a \to a_t$, ϕ an invariant state ($\phi \circ \tau_t = \phi$) and Π_ϕ the associated representation in $\mathscr{H}_\phi$. We call $(\mathscr{A}, \tau, \phi)$ strongly (resp. weakly) asymptotically Abelian if $[\Pi_\phi(a), \Pi_\phi(b_t)]$ converges strongly (resp. weakly) to zero $\forall a$, $b \in \mathscr{A}$, $t \to \pm\infty$.

Remark 2.3. Without reference to Π_ϕ the conditions can be equivalently stated as

$$\phi(c[a^*, b_t^*][a, b_t]\, d) \xrightarrow{t \to \infty} 0 \text{ resp. } \phi(c[a, b_t]\, d) \xrightarrow{t \to \infty} 0 \quad \forall a, b, c, d \in \mathscr{A}$$

Independent systems are tensor products and an invariant state which has product structure for widely separated parts leads to an asymptotic Abelian situation.

Definition 2.4. An invariant state ϕ is called hyperclustering (resp. clustering) if for $t \to \infty$

$$\phi(ab_t c\, d_t e) \to \phi(ace)\, \phi(bd) \text{ resp. } \phi(ab_t c) \to \phi(ac)\, \phi(b) \quad \forall a, bc, d, e \in \mathscr{A}$$

Proposition 2.5. If ϕ is hyperclustering (resp. clustering), then $(\mathscr{A}, \tau, \phi)$ is strongly (resp. weakly) asymptotically Abelian.

Proof. Using Remark 2.3, we see for the strong properties

$$\begin{aligned} \phi(c[a^*, b_t^*][a, b_t]\, d) &= \phi(ca^*b_t^*ab_t d) + \phi(cb_t^*a^*b_t ad) \\ &\quad - \phi(ca^*b_t^*b_t ad) - \phi(cb_t^*a^*ab_t d) \\ &\to 2\phi(ca^*ad)\, \phi(b_t^*b_t) \\ &\quad - 2\phi(ca^*ad)\, \phi(b_t^*b_t) = 0 \end{aligned}$$

and even more trivially for the weak properties.

Remarks 2.6.

1. By Riemann–Lebesgue the thermal correlation functions (2.1) decay both for the shift $\tilde{\alpha}(k) \to e^{ikx}\tilde{\alpha}(k)$ and the free time evolution $\tilde{\alpha}(k) \to e^{-ik^2t}\tilde{\alpha}(k)$ such that ϕ_β, $0 \leqslant \beta \leqslant \infty$, is strongly clustering and $\mathscr{A}$ is strongly

asymptotically Abelian for these automorphisms. For ϕ_0 and ϕ_∞ the same is true for the boost $\tilde{\alpha}(k) \to \tilde{\alpha}(k+p)$.

2. Since hyperclustering implies that the commutator applied to any fixed vector $a\,|\varphi\rangle \in \mathscr{H}_\phi$ goes to zero, we see that all higher n-point functions factor automatically:

$$\begin{aligned}\phi(a^{(1)}b_t^{(1)}a^{(2)}b_t^{(2)}\cdots a^{(n)}) &= \langle\varphi|\, a^{(1)}[b_t^{(1)}, a^{(2)}]\cdots a^{(n)}\,|\varphi\rangle \\ &\quad + \langle\varphi|\, a^{(1)}a^{(2)}b_t^{(1)}\cdots a^{(n)}\,|\varphi\rangle = \cdots \\ &= \phi(a^{(1)}a^{(2)}\cdots a^{(n)}b_t^{(1)}\cdots b_t^{(n-1)}) \\ &\quad + \text{terms going to zero for } t\to\pm\infty\end{aligned}$$

3. Clustering and strong asymptotic Abeliannes together imply hyperclustering.

In a mixing classical system each part of phase space gets in the course of time finely dispersed over the whole phase space. Correspondingly, each subvolume U eventually overlaps with each fixed volume V such that the product of the characteristic functions $\chi_{U_t}\cdot\chi_V$ is 1 at some places $\forall t > T$. This observation lends itself to a definition of mixing for an arbitrary C^*-dynamical system. Again we need a distinction without classical analogue.

Definition 2.7. A dynamical system $(\mathscr{A}, \tau)$ is called strongly (resp. weakly) mixing if, $\forall a, b \in \mathscr{A}$,

$$\lim_{t\to\infty} \|ab_t\| = \|a\|\,\|b\|$$

(resp. $\exists T$ such that $ab_t \neq 0$, $\forall t > T$).

Remark 2.8. For a classical system where $\mathscr{A} = C(M)$, the continuous functions over the phase space M, and τ is generated by homeomorphisms $\tau_*: M \to M$, $\tau(f(x)) = f(\tau_* x)$ weak and strong mixing are equivalent to what is called topologically mixing. It is defined as follows: $\forall U, V \subset M$ and open $\exists T$ such that $\tau_{*t}(U) \cap V \neq 0$, $\forall t > T$. It implies in particular the existence of a dense orbit and that time-invariant functions $\in C(M)$ are constant on M. In Definition 2.7 we do not use the adjective "topological." It seems redundant, since there is no other structure mixing can refer to.

Theorem 2.9. A dynamical system $(\mathscr{A}, \tau)$ is strongly (resp. weakly) mixing if it has a hyperclustering (resp. clustering) invariant state ϕ with Π_ϕ (resp. ϕ) faithful.

Proof.

(i) The weak properties: $\phi(ab_t b_t^* a^*) \to \phi(aa^*)\,\phi(bb^*) > 0$. Thus, $\exists T$ such that $\phi(ab_t b_t^* a^*) > 0$ and therefore $ab_t \neq 0$, $\forall t > T$.

(ii) The strong properties: Since $cd_t ab_t b_t^* a^* d_t^* c^* \leqslant \|ab_t\|^2\, cd_t d_t^* c^*$, we have, $\forall a, b, c, d \in \mathscr{A}$,

$$\frac{\phi(cd_t ab_t b_t^* a^* d_t^* c^*)}{\phi(cd_t d_t^* c^*)} \leqslant \|ab_t\|^2$$

By the cluster property the left-hand side tends for $t \to \infty$ to

$$\frac{\phi(caa^*c^*)\,\phi(dbb^*d^*)}{\phi(cc^*)\,\phi(dd^*)}$$

and faithfulness of Π_ϕ implies

$$\sup_{c \in \mathscr{A}} \frac{\phi(caa^*c^*)}{\phi(cc^*)} = \|a\|^2$$

Thus,

$$\lim_{t \to \infty} \|ab_t\|^2 \geqslant \|a\|^2\, \|b\|^2$$

but generally $\|ab_t\| \leqslant \|a\|\, \|b\|$, which proves Theorem 2.9.

Remark 2.10. In ref. 2, strong mixing is proved without reference to a state provided that $\mathscr{A}$ is simple (which is the case for UHF-algebras) and Definition 2.2 is strengthened to $\|[a, b_t]\| \to 0$. This norm convergence is satisfied for the even elements of the Fermi field algebra under translations or free time evolution, but hard to prove for realistic interactions.

3. CLUSTER PROPERTIES OF Φ_0 and Φ_∞ FOR TIME EVOLUTIONS WITH GALILEAN-INVARIANT INTERACTIONS

In the last section we noted that ϕ_0 and ϕ_∞ are hyperclustering for the shifts in configuration and momentum space and free time evolution. We shall now examine what remains true for interactions which are invariant under the first two automorphisms. First we have to inspect the structure of the group generated by these various transformations.

Definition 3.1. The automorphism groups shift σ_ρ, boost γ_ρ, and gauge transformation v_λ are defined by actions on α_f as follows:

$$\sigma_\rho(\alpha_{f(x)}) = \alpha_{f(x+\rho)}, \qquad \gamma_p(\alpha_{\tilde{f}(k)}) = \alpha_{\tilde{f}(k+p)}, \qquad v_\lambda(\alpha_f) = \alpha_{e^{i\lambda}f}.$$

They satisfy

$$\sigma_\rho \circ \nu_\lambda = \nu_\lambda \circ \sigma_\rho, \quad \gamma_p \circ \nu_\lambda = \nu_\lambda \circ \gamma_p, \quad \gamma_p \circ \sigma_\rho = \sigma_\rho \circ \gamma_p \circ \nu_{-p\rho}$$

A time evolution τ_t is called gauge and Galilean invariant if

$$\tau_t \circ \nu_\lambda = \nu_\lambda \circ \tau_t, \quad \tau_t \circ \sigma_\rho = \sigma_\rho \circ \tau_t, \quad \tau_t \circ \gamma_p = \gamma_p \circ \tau_t \circ \sigma_{-tp} \circ \nu_{-tp^2/2}$$

Remarks 3.2.

1. The properties of τ_t are abstracted from finite systems with a Hamiltonian which consists of a kinetic energy $p^2/2$ and a potential invariant under σ, γ, and ν. For an infinite system such a Hamiltonian with a local potential could be formally written as

$$\frac{1}{2} \int d^3x\, \nabla\alpha^*(x)\, \nabla\alpha(x)$$
$$+ \int dx\, dx'\, \alpha^*(x)\, \alpha^*(x')\, v(x - x')\, \alpha(x')\, \alpha(x)$$

but it remains to be studied how to interpret such an expression.

2. In the representations Π_{ϕ_0} and Π_{ϕ_∞} these automorphisms are unitarily implemented, since ϕ_0 and ϕ_∞ are invariant under their action. In Π_{ϕ_β}, $0 < \beta < \infty$, the Galilei group is broken and G does not exist. Denoting the unitaries corresponding to ν_λ, σ_ρ, τ_t, and γ_p by $e^{i\lambda N}$, $e^{iP\rho}$, e^{-iHt}, and e^{ipG}, respectively, we deduce the commutation relations

$$[P, G] = iN, \quad [H, G] = iP, \quad [G, N] = [P, N] = [H, N] = 0 \tag{3.1}$$

The representation theory of such an algebra is well known, since we know that N is in the center and has integer eigenvalues $n \in \mathbf{Z}$. We have to distinguish between $n = 0$, where P and G commute, and $n \neq 0$, where they satisfy Heisenberg's commutation relations. Thus, in Hilbert space we have the following sectors (compare, e.g., ref. 7):

(a) The $n \neq 0$ sector. The operators can be written

$$P = p \otimes \mathbf{1}, \quad G = \frac{n}{i} \frac{\partial}{\partial p} \otimes \mathbf{1}, \quad H = \frac{1}{2n} p^2 \otimes \mathbf{1} + \mathbf{1} \otimes H_{\text{int}} \tag{3.2}$$

where H_{int} means some internal Hamiltonian about which we cannot say anything. However, in the first factor the Hilbert space is $L^2(\mathbf{R}, dp)$ such that the spectrum of p^2 is absolutely continuous.

(b) The $n=0$ sector. This sector contains the cyclic vector $|\Omega\rangle$ corresponding to ϕ_0, resp. ϕ_∞. They are eigenvectors for all four generators, all of them having eigenvalue zero. The representations of P, G, and N are explicitly known and show that there is no other invariant vector. In the rest their action is of the form

$$P=p\otimes \mathbf{1}, \qquad G=p\otimes i\frac{\partial}{\partial h}, \qquad H=\mathbf{1}\otimes h \tag{3.3}$$

Here we know nothing about p, but the second factor in the Hilbert space is $L^2(\mathbf{R}, h)$ and thus H has an absolutely continuous spectrum.

Proposition 3.3. For $t\to\pm\infty$, e^{iHt} converges weakly to $|\Omega\rangle\langle\Omega|$.

Proof. From the representations (3.2) and (3.3) we see that apart from $|\Omega\rangle$, H has an absolutely continuous spectrum. Thus, Proposition 3.3 follows from Riemann–Lebesgue.

Corollary 3.4. $(\mathscr{F}, \tau, \phi_0)$ is clustering and therefore weakly asymptotically Abelian and weakly mixing.

Proof. $\phi_0(ab_tc)=\phi_0(cab_t)=\langle\Omega|\, cae^{-iH't}b\,|\Omega\rangle\to\phi_0(ca)\,\phi_0(b)$.

Remark 3.5. In this case the same conclusion holds even for $\Pi_{\phi_0}(\mathscr{F})''$, the strong closure of $\Pi_{\phi_0}(\mathscr{F})$. The above argument does not work for ϕ_∞, since it does not have the cyclic properties of ϕ_β, $\beta<\infty$. Thus, the clustering property of $\phi_\infty(ab_t)$ does not extend generally to $\phi_\infty(ab_tc)$. Actually, $(\Pi_{\phi_\infty}(\mathscr{F})'', \sigma, \phi_\infty)$ is not asymptotically Abelian, since $\Pi_{\phi_\infty}(\mathscr{F})''$ contains noncommuting translation invariant elements.

Finally, we shall inspect the mixing properties in Π_{ϕ_∞}. One has the feeling that in this representation $\tau_t(\alpha_f^*)$ should converge weakly to zero for $t\to\pm\infty$, since it creates a particle far away and these vectors become orthogonal to the others in $\mathscr{H}_{\phi_\infty}$ which live mainly in finite regions. Similarly, $\tau_t(\alpha_f)$ should converge strongly to zero, since it wants to destroy particles where there are none. Finally, $[\tau_t(\alpha_f), \alpha_g^*]_+$ may be expected to converge in norm to zero since $\tau_t(\alpha_f)$ tries to destroy the particle in the fixed state g and this ought to go to zero for $t\to\pm\infty$ irrespective to the vector to which one applies this procedure.

To see what can be substantiated by proofs, we start the following results.

Proposition 3.6. $\tau_t(\Pi_{\phi_\infty}(\alpha_f))$ converges strongly to zero for $t\to\pm\infty$.

Proof. In Π_{ϕ_∞} the $n=0$ sector consists only of the vacuum $|0\rangle$ and since $\alpha_f|0\rangle=0$, we can turn to the $n>0$ sectors (3.2). Here

$$\mathscr{H}_{\phi_\infty}=\mathscr{H}_v\otimes\mathscr{H}_w, \qquad \psi=v\otimes w$$
$$U_t=\exp[-iP^2t/2n]\otimes\exp[-iH_It],, \qquad U_t\psi=v_t\otimes w_t$$
$$v_t=\exp[-iP^2/t/2n]v\rightharpoonup 0$$

This implies that for any projection $q\in\mathscr{B}(\mathscr{H}_v)$ with $\mathrm{tr}_{\mathscr{H}_v}q<\infty$ and $\varepsilon>0$, $\exists T$ such that $\|qv_t\|<\varepsilon$, $\forall t>T$. Furthermore, $\forall A\in\mathscr{B}(\mathscr{H}_{\phi_\infty})$ with $\|\mathrm{tr}_{\mathscr{H}_v}A^*A\|<\infty$, where $\mathrm{tr}_{\mathscr{H}_v}$ is now the partial trace, there exists a projection $q\in\mathscr{B}(\mathscr{H}_v)$ with $\mathrm{tr}_{\mathscr{H}_v}q<\infty$ and

$$\|(1-Q)A^*A(1-Q)\|<\varepsilon^2\Leftrightarrow\|A(1-Q)\|<\varepsilon \qquad \text{where} \quad Q=q\otimes\mathbf{1}$$

For such an A we have $AU_t\psi\to 0$, since

$$\|AU_tv\otimes w\|=\|A(1-Q+Q)v_t\otimes w_t\|$$
$$\leqslant\|A(1-Q)\|+\|A\|\ \|qv_t\|\ \|w_t\|<2\varepsilon, \qquad \forall t>T$$

and the vectors of the form $v\otimes w$ are total in $\mathscr{H}_{\phi_\infty}$.

To apply these results to α_f, we note that $\alpha_f^*\alpha_f$ acts in the n-particle sector in a momentum representation as the integral kernel

$$K(p_1\cdots p_n;p_1'\cdots p_n')=\sum_{j=1}^{n}\delta(p_1-p_1')\cdots f(p_j)$$
$$\times f^*(p_j')\cdots\delta(p_n-p_n')$$

The partial trace over the total momentum is an operator in the space with $\sum_j p_j=0$ and is given by

$$\mathrm{tr}_{\mathscr{H}_v}\alpha_f^*\alpha_f\leftrightarrow\int dqK(p_1+q,\dots,p_n+q;p_1'+q,\dots,p_n'+q)$$
$$=n\prod_{j=1}^{n-1}\delta(p_j-p_j')\int dq\,|f(q)|^2$$
$$\leftrightarrow\mathbf{1}\cdot n\int dq\,|f(q)|^2$$

This is not a bounded operator in $\mathscr{H}_w$, but it is dominated by $N=\int dp\,\tilde{\alpha}^*(p)\,\tilde{\alpha}(p)$. Thus, $\alpha_f/(1+N)$ satisfies the criteria for A and since

$(1+N)^{-1}$ is a bijection $\mathscr{H}_w \leftrightarrow \mathscr{H}_w$ we can argue that also the $\varphi = [1/(1+N)]v \otimes w$ are total in $\mathscr{H}_{\phi_\infty}$. Thus,

$$\|\alpha_t \varphi\| = \left\| \alpha U_t \frac{1}{N+1} v \otimes w \right\|$$
$$= \left\| \alpha \frac{1}{1+N} v_t \otimes w_t \right\| \to 0$$

proves Proposition 3.6.

Corollary 3.7. $\tau_t(a)$ converges weakly to $\phi_\infty(a)\ \forall a \in \Pi_{\phi_\infty}(\mathscr{F})$; it does so strongly for $a \in \Pi_{\phi_\infty}(\mathscr{A})$.

Remark 3.8. By $\Pi_{\phi_\infty}(\mathscr{A})$ we mean the representation of $\mathscr{A}$ in $\mathscr{H}_{\phi_\infty}$, the Hilbert space for $\Pi_{\phi_\infty}(\mathscr{F})$, and not the one-dimensional representation $a \to \phi_\infty(a)$ which one gets from the GNS construction for $\mathscr{A}$ with ϕ_∞.

Proof. Since $A \to A^*$ is weakly (but not strongly) continuous, $\tau_t(\alpha_f^*)$ converges weakly to zero [not strongly as $\tau_t(\alpha_f \alpha_f^*) \to \phi_\infty(\alpha_f \alpha_f^*) = \|f\|^2 \neq 0$]. Using the CAR, any $a \in \Pi_{\phi_\infty}(\mathscr{F})$ can be ordered,

$$a = \phi_\infty(a) + \sum \alpha_{f_1}^* \alpha_{f_2}^* \cdots \alpha_{f_n}^* \alpha_{g_1} \cdots \alpha_{g_m} \tag{3.4}$$

Now, for $t \to \pm\infty$ each term with $m \neq 0$ in $\sum$ converges strongly to zero, the others only weakly.

Corollary 3.9. $(\Pi_{\phi_\infty}(\mathscr{F}),\ \tau,\ \phi_\infty)$ is weakly asymptotically Abelian and clustering.

Proof. $ab_t - b_t a$ converges weakly to $a\phi(b) - \phi(b)a = 0\ \forall a, b \in \Pi_{\phi_\infty}(\mathscr{F})$. Thus,

$$\phi_\infty(ab_t c) \to \phi_\infty(acb_t) \to \langle 0|\ ac\ |0\rangle\langle 0|\ b\ |0\rangle$$

Corollary 3.10. $(\Pi_{\phi_\infty}(\mathscr{A}),\ \tau,\ \phi_\infty)$ is strongly asymptotically Abelian and hyperclustering.

Proof. For $a, b \in \mathscr{A}$, $ab_t - b_t a$ converges strongly to $a\phi(b) - \phi(b)a = 0$. The hyperclustering follows from Remark 2.6.3.

Remarks 3.11.

1. The strong conclusion does not hold for $(\Pi_{\phi_\infty}(\mathscr{F}),\ \tau,\ \phi_\infty)$ and therefore we cannot conclude mixing properties for any of the two algebras. $\mathscr{A}$ is not simple; in fact, the a's of the form in Corollary 3.7 with $\phi_\infty(a) = 0$ are a two-sided ideal I of $\mathscr{A}$. The GNS construction with ϕ_∞ of $\mathscr{A}$

represents faithfully only $\mathscr{A}\backslash I$ by the one-dimensional representation $a \to \phi_\infty(a)$. On the positive cone generated by the elements of the form $\alpha_{f_1} \cdots \alpha_{f_n} \alpha^*_{f_n} \cdots \alpha^*_{f_1}$ we have

$$\phi_\infty(a) = \|f_1\|^2 \, \|f_2\|^2 \cdots \|f_n\|^2 = \|a\|$$

and thus for them the canonical map $\mathscr{A} \to \mathscr{A}\backslash I$, $a \to \phi_\infty(a)$ is an isometry. Then the conclusion of Theorem 2.4 holds and we have strong mixing, but only on this positive cone. On the other hand, ϕ_∞ is not faithful on $\mathscr{F}$; thus, Theorem 2.9 does not apply.

2. Our proof applies to start with two polynomials in the field operators. It is readily seen to extend to the norm closure of this algebra, that is, the C^*-algebras $\Pi_{\phi_\infty}(\mathscr{A})$ and $\Pi_{\phi_\infty}(\mathscr{F})$; respectively. It does not, however, extend to the von Neumann algebras $\Pi_{\phi_\infty}(\mathscr{A})''$ and $\Pi_{\phi_\infty}(\mathscr{F})''$, since the limits $t \to \pm\infty$ will not commute with the strong closure.

We can show the strong version of Abelianness, clustering, and mixing for $\mathscr{A}$ only if in addition to Galilei invariance we impose some short-range condition on the interaction. It is the following.

Definition 3.12. Denote by H_n the restriction of the Hamiltonian in $\Pi_{\phi_\infty}(\mathscr{F})$ to the n-particle sector such that $H_1 = H_0$ is the free Hamiltonian. Write $H_n = H_{n-1} + H_1 + V_1$ and $V_1(t) = \exp[iH_1 t]\, V_1 \exp[-iH_1 t]$. We say that $(\tau, \mathscr{F}, \phi_\infty)$ has asymptotically trivial Møller operators if

$$s\text{-}\lim_{t \to \pm\infty} \exp\{-it[H_{n-1} + H_1 + V_1(t)]\} \exp[it(H_{n-1} + H_1)] = \mathbf{1}$$

Remark 3.13. For local pair potentials,

$$V_1(t) = \sum_{j=2}^{n} V(x_1(t) - x_j)$$

and asymptotic triviality means that the Møller transformation approaches unity at infinity in configuration space. We shall not determine the exact class of potential for which this is true, but give in the Appendix some indications where it holds.

Theorem 3.14. A Galilean-invariant system with asymptotically trivial Møller operator $(\Pi_{\phi_\infty}(\mathscr{A}), \tau, \phi_\infty)$ is strongly asymptotically Abelian, hyperclustering, and strongly mixing. For $\Pi_{\phi_\infty}(\mathscr{F})$ the same conclusion holds provided the first two properties are understood with the appropriate signs to accommodate anticommutativity.

Proof. We start with asymptotic Abelianness and first note that it is sufficient to show that $s\text{-lim}_{t\to\pm\infty}[\alpha_g^*, \tau_t(\alpha_f^*)]_+ = 0$. Second, we remark that this is equivalent to $[\alpha_{g_1}^* \cdots \alpha_{g_n}^*, \tau_t(\alpha_f^*)]_\pm |0\rangle \to 0$ since a vector in the n-particle sector can be written $A_n^* |0\rangle$ with

$$A_n^* = \int dx_1 \cdots dx_n \psi_n(x_1 \cdots x_n)\, \alpha^*(x_1) \cdots \alpha^*(x_n)$$

and

$$(\alpha_f^* \alpha_g^* + \alpha_g^* \alpha_f^*) A_n^* = \alpha_f^* \alpha_g^* A_n^* + (-)^n \alpha_g^* A_n^* \alpha_f^* \\ + \alpha_g^*(\alpha_f^* A_n^* + (-)^{n-1} A_n^* \alpha_f^*)$$

Now

$$(U_t^{-1} \alpha_f^* U_t A_{n-1}^* + (-)^n A_{n-1}^* U_t^{-1} \alpha_f^* |0\rangle$$

has the wave function

$$e^{-iH_n t} \sum_{j=1}^{n} f_j(x_1)(-)^j e^{iH_{n-1}t} \psi_{n-1}(x_1,..., \hat{x}_j, x_n) \\ - \sum_{j=1}^{n} (-)^j f_{-t}(x_j)\, \psi_{n-1}(x_1,..., \hat{x}_j,..., x_n)$$

($\hat{x}_j$ means that this argument is deleted.) Such a vector converges strongly to zero iff

$$e^{iH_1 t} e^{-iH_n t} e^{iH_{n-1} t} \to 1$$

Since H_1 and H_{n-1} commute, they act on different variables; this is equivalent to

$$e^{iH_1 t} e^{-iH_n t} e^{-iH_1 t} e^{i(H_{n-1} + H_1)t} \to 1$$

which is exactly the asymptotic trviality. This proves strong asymptotic Abeliannes, which, together with $U_t \to |0\rangle\langle 0|$, implies hyperclustering

$$\langle 0| ab_t cd_t e |0\rangle \to \langle 0| aceb_t d_t |0\rangle \\ = \langle 0| aceU_t bd |0\rangle \to \langle 0| ace |0\rangle\langle 0| bd |0\rangle$$

Appealing to Theorem 2.4, we get the strong mixing.

Summarizing our findings, we state the following result.

Theorem 3.15. Let τ_t be a gauge- and Galilei-invariant time evolution of the fermion algebra $\mathscr{A}$. Then $(\mathscr{A}, \tau)$ is weakly mixing. If in addition

$(\mathscr{A}, \tau, \phi_\infty)$ has asymptotically trivial Møller operators, then it is even strongly mixing.

Remark 3.16. Although the free time evolution has both properties, there is no mixing of the algebra $\mathscr{A}|_n$ reduced to a sector with a fixed number n of particles. For instance, for one particle $\mathscr{A}|_1$ are the compact operators in $L^2(\mathbf{R}^3)$ and the product of projectors with disjoint support in momentum space remains always zero if one of them evolves under the free time evolution.

APPENDIX

Here we discuss when we can expect that the Møller operators are asymptotically trivial. We do not intend to present the explicit analysis, but show the connection of the problem with the usual considerations to prove the existence and completeness of the Møller operators (see refs. 8–10).

The existence of the limit is less problematic; thus, we might as well consider the Cesaro limit. Therefore, we want to show

$$\begin{aligned}
1 &= \text{st-}\lim_{t\to\infty} \exp\{-it[H_{n-1} + H_1 + V_{1,n-1}(t)]\} \\
&\quad \times \exp[it(H_{n-1} + H_1)] \\
&= \text{w-}\lim_{t\to\infty} \exp\{-it[H_{n-1} + H_1 + V_{1,n-1}(t)]\} \\
&\quad \times \exp[it(H_{n-1} + H_1)] \\
&= \text{w-}\lim_{\varepsilon\searrow 0} \varepsilon \int_0^\infty \exp(itH_1) \exp[-it(H_{n-1} + H_1 + V_{1,n-1})] \\
&\quad \times \exp(itH_{n-1} - \varepsilon t)\, dt
\end{aligned}$$

It suffices to show the weak convergence on a dense set. We have used the formulation with the time-dependent potential because it is more suggestive, since we know that $V_{1,n-1}(t)$ converges strongly to zero. But for calculation we concentrate on the last form

$$\begin{aligned}
1 = \text{w-}\lim_{\varepsilon\searrow 0} \int dE_1\, dE_2\, \delta(H_1 - E_1) \\
\times \frac{-i\varepsilon}{H_n - E_1 - E_2 - i\varepsilon}\, \delta(H_{n-1} - E_2)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \lim_{\varepsilon \searrow 0} \int dE_1\, dE_2\, \delta(H_1 - E_1) \\
&\quad \times \frac{i\varepsilon}{H_1 + H_{n-1} - E_1 - E_2 - i\varepsilon} V_{1,n-1} \\
&\quad \times \frac{1}{H_n - E_1 - E_2 - i\varepsilon} \delta(H_{n-1} - E_2) \\
&= 1 - \lim_{\varepsilon \to 0} \varepsilon \int dE_1\, dE_2\, \delta(H_1 - E_1) \frac{1}{H_{n-1} - E_2 - i\varepsilon} \\
&\quad \times V_{1,n-1} \frac{1}{H_1 - E_1 - i\varepsilon} \delta(H_{n-1} - E_2) \\
&\quad - \lim_{\varepsilon \to 0} \varepsilon \int dE_1\, dE_2\, \delta(H_1 - E_1) \frac{1}{H_{n-1} - E_2 - i\varepsilon} \\
&\quad \times V_{1,n-1} \frac{1}{H_n - E_1 - E_2 - i\varepsilon} V_{1,n-1} \frac{1}{H_1 - E_1 - i\varepsilon} \delta(H_{n-1} - E_2)
\end{aligned}
$$

It remains to argue that the integral gives a densely defined form. In this case the expectation value with a dense set is finite and thus $\lim_{\varepsilon \to 0} \varepsilon \langle \cdots \rangle$ gives zero. For $n = 2$ we have for the first term the integral kernel in the spectral representation of the momenta

$$
\frac{1}{p_2^2 - p_2'^2 - i\varepsilon} V(p_1 - p_1' - p_2 + p_2') \frac{1}{p_1^2 - p_1'^2 - i\varepsilon}
$$

which corresponds to a bounded operator on weighted L^2 spaces for sufficiently regular potentials.[8] The proofs of the existence of the Møller operators should also give a control on the type of the singularity of the kernel

$$
\begin{aligned}
&\langle E_1, E_2' | \frac{1}{H_{n-1} - E_2 - i\varepsilon} V_{1,n-1} \frac{1}{H_n - E_1 - E_2 - i\varepsilon} \\
&\quad \times V_{1,n-1} \frac{1}{H_1 - E_1 - i\varepsilon} | E_1', E_2 \rangle
\end{aligned}
$$

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